The optimal prize structure of symmetric Tullock contests*

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Abstract

We show that the optimal prize structure of symmetric n-player Tullock tournaments assigns the entire prize pool to the winner, provided that a symmetric pure strategy equilibrium exists. If such an equilibrium fails to exist under the winner-take-all structure, we construct the optimal prize structure which improves existence conditions by dampening efforts. If no such optimal equilibrium exists, no symmetric pure strategy equilibrium induces positive efforts. (JEL *C7, D72, J31.* Keywords: *Tournaments, Incentive structures, Rent seeking.*)

1 Introduction

It is well known that "an income maximizing contest administrator obtains the most rent-seeking contributions when he makes available a single, large prize" Clark and Riis (1998b). Less, however, is known about effort maximizing prizes in Tullock contests when an equilibrium supporting this winner-take-all structure does not exist or if non-linear costs accompany the outlays of more than two contestants. Unfortunately, both these cases typically arise in practical applications. We show that with symmetric players, the winner-take-all prize structure induces maximal efforts regardless of the number of players or their effort cost, provided that a symmetric pure strategy equilibrium exists. In cases where such an equilibrium fails to exist under the winner-take-all prize structure, we construct optimal prizes which improve existence conditions by dampening excessive efforts. If no such equilibrium exists, no symmetric pure strategy equilibrium induces positive efforts. As any optimal equilibrium leaves zero utility to

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the contestants in order to maximize efforts, the highest achievable equilibrium effort is the same for all symmetric equilibria. This aspect of our results resembles what was shown by Barut and Kovenock (1998) for the fully discriminating, complete information all-pay auction (which is the limit case of our setup). Since utility is zero for any effort choice in the support of their mixed equilibria, they derive—in contrast to our results—the near total arbitrariness of prize structures. Adding incomplete information to the all-pay auction setup, Moldovanu and Sela (2001) show that more than one prize is optimal when contestants have convex costs. It may thus come as a surprise that our optimal prize structures are independent of the curvature of costs. The reason for this disparity is that their heterogenous players have private effort costs which affect bidding behavior. Intuitively, a second prize in an asymmetric contest can be desirable for the maximization of total efforts because a single first prize may undermine the incentives of both weak contestants expecting not to win and of strong contestants believing to be able to win with little effort. Szymanski and Valletti (2005) confirm this intuition in an asymmetric Tullock contest. Related studies concerned with multi-stage optimal prizing are Gershkov and Perry (2008) and Fu and Lu (2006). The former analyze the effort effects of introducing midterm reviews into the contest while the latter show the optimality of a single prize at each stage of their multi-stage Tullock tournament (with linear cost) in a parameter range where existence is not problematic.[1] We show that, as such, multiple prizes are not optimal in symmetric Tullock contests but multi-prize configurations may be attractive in order to *dampen* incentives to obtain equilibrium existence when equilibria do not exit under the winner-take-all configuration.

2 Model and results

We consider a set $\mathcal{N}$ of $n > 1$ symmetric, risk neutral players engaging in a contest where any player $i \in \mathcal{N}$ exerts effort $e_i \in [0, \infty)$. There is a fixed prize pool $P > 0$ from which prizes $P^1, P^2, \ldots, P^n$, $P = \sum_l P^l$, awarded to the contest winner, second etc. are taken. The contest satisfies limited liability and the designer sets $P^l \geq 0$, $l = 1, \ldots, n$ in order to maximize the sum of efforts. Denote the vector of all players' efforts by $\hat{e} = (e_1, e_2, \ldots, e_n)$. Then the winning probability of player i exerting effort e_i with her opponents choosing $\hat{e}_{-i}$ is given by the Tullock success function as[2]

$$f_i^1(\hat{e}) = \frac{e_i^r}{\sum_{j \in \mathcal{N}} e_j^r} \text{ for } r > 0.$$

We define $f_i^1(0) = 1/n$ for completeness. The probabilities of winning the second, third prize etc. $f_i^2, f_i^3, \ldots$ are given by the nested Tullock success function, i.e. by recursively applying the above success function to the set of players without the winners of the previous stages. Hence player i

[1] A recent and comprehensive review of the tournaments literature including the Tullock contest is Konrad (2008). It allows us to omit all but the most relevant references here.

[2] Skaperdas (1996) argues that the Tullock form is less special than one might believe. In particular, he shows that it is the only ratio-based function fulfilling a small set of intuitive desiderata.

chooses her effort e_i in order to maximize her utility

$$\arg\max_{e_i} \sum_{l=1}^{n} \left(f_i^l(e_i, \hat{e}_{-i}) P^l \right) - c(e_i) \tag{1}$$

where we assume $c(e_i)$ to be monotonic. Assuming the existence of a symmetric equilibrium in pure strategies, our first result shows that the winner-take-all structure induces the highest efforts for arbitrary costs.[3] All proofs can be found in the appendix.

Proposition 1. *Given equilibrium existence, the Tullock tournament which induces the highest sum of equilibrium efforts from symmetric contestants assigns the whole prize pool to the winner.*

In the second proposition we generalize Clark and Riis (1998b) in deriving an equilibrium existence condition for the winner-take-all prize structure. From now on, we restrict attention to cost functions of the form $c(e) = ae^b$ with $a, b > 0$ for expositional simplicity.[4]

Proposition 2. *Existence of a symmetric pure strategy equilibrium under the winner-take-all prize structure $P^1 = P$ is ensured if and only if*

$$\frac{r}{b} \leq \frac{n}{n-1}. \tag{2}$$

We now analyze the optimal symmetric pure strategy equilibrium in cases where the winner-take-all prize structure $P^1 = P$ causes excessive efforts destroying the equilibrium. We show that a more evenly distributed prize structure dampens efforts and extends the range of parameters where existence can be obtained.

Proposition 3. *There is a monotonic prize structure for which a symmetric pure strategy equilibrium inducing positive efforts exists, if and only if*[5]

$$\frac{r}{b} \leq \frac{n-1}{\sum_{k=2}^{n} \frac{1}{k}}. \tag{3}$$

The above proposition identifies a prize structure which ensures equilibrium existence. The next proposition shows that the following (similar) prize structure is also optimal: The highest possible effort in any symmetric equilibrium is $e^* = c^{-1}(P/n)$. Given that (3) is satisfied, the designer can implement maximal equilibrium efforts e^* by trying first $P^1 = P$, then $P^1 = P^2 = P/2$, then $P^1 = P^2 = P^3 = P/3$ and so forth until the resulting efforts $\tilde{e}$ eventually sink below e^*. For the first such uniform prize structure he then shifts some $\varepsilon > 0$ away from the last prize k and subdivides

[3] We concentrate attention on symmetric equilibria in pure strategies. Alternatives are discussed, among others, by Baye, Kovenock, and de Vries (1994), Szymanski and Valletti (2005), and Cornes and Hartley (2005).

[4] The analysis can be done for more general cost but then no explicit existence threshold values can be derived.

[5] Since (2) and (3) coincide for $n = 2$, existence conditions cannot be improved in this well known case.

it equally among the $k-1$ prior prizes until the efforts $\tilde{e}$ exactly equal e^*. The following proposition formalizes this idea.

Proposition 4. *For $n \geq 3$, if (2) is violated but (3) holds, then there exists an integer $2 \leq k < n$ and a real number $0 \leq \varepsilon < \frac{1}{k}P$ such that the prize structure*

$$(\underbrace{\frac{1}{k}P + \frac{1}{k-1}\varepsilon, \frac{1}{k}P + \frac{1}{k-1}\varepsilon, \ldots, \frac{1}{k}P + \frac{1}{k-1}\varepsilon}_{k-1 \text{ times}}, \frac{1}{k}P - \varepsilon, 0, \ldots, 0) \tag{4}$$

is optimal, i.e. induces efforts $e^* = c^{-1}(P/n) = \left(\frac{1}{a}\frac{P}{n}\right)^{\frac{1}{b}}$.

3 Discussion

No symmetric equilibrium inducing positive efforts exists for monotonic prizes if (3) fails. If the designer benefits from retaining part of the prize pool and an equilibrium exists, he can balance this prize reduction with the lower extracted efforts without affecting existence. The following picture illustrates the interplay of the above propositions. It shows the utility $U_i(e_i; e^*)$ of a player unilaterally deviating

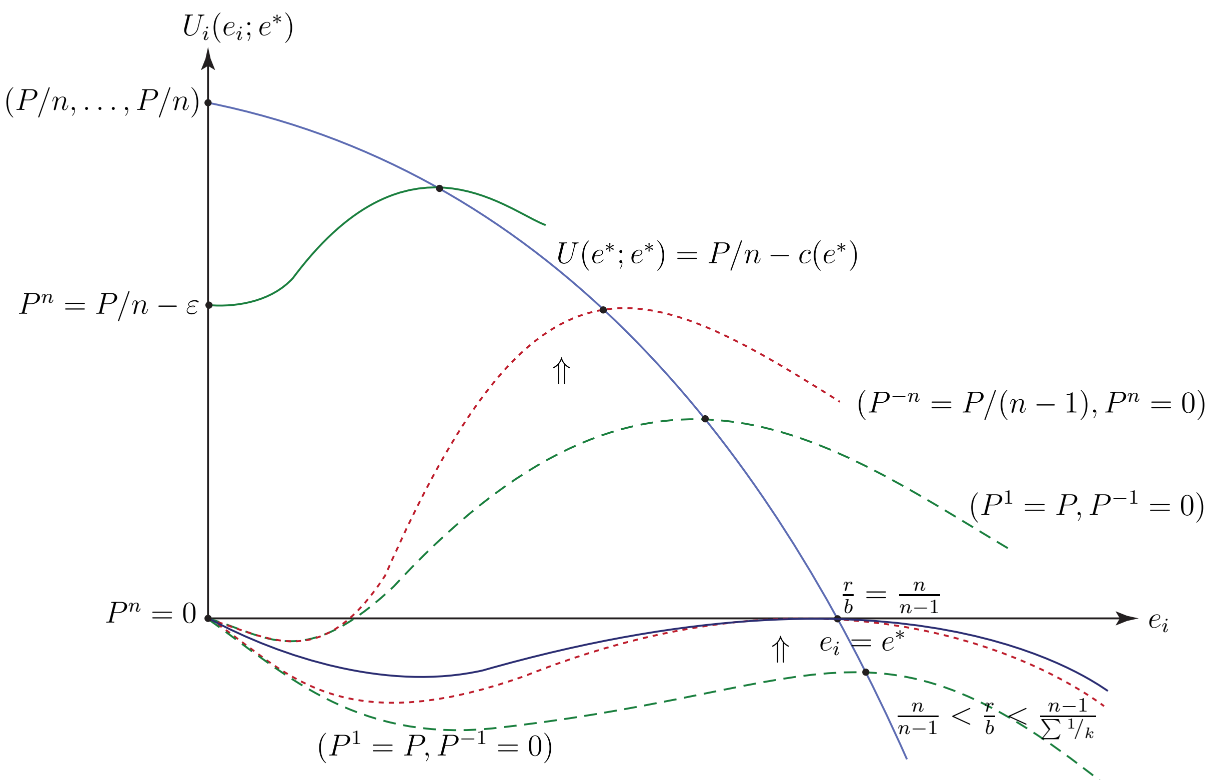

Figure 1: Unilateral deviations from symmetric equilibrium for different prize structures and values of r/b.

from e^*. The blue downward sloping curve depicts the locus of possible symmetric equilibria where utility is given by $P/n - c(e^*)$. There are two things to note: First, softening incentives through suboptimal multi-prize structures increases the players' equilibrium utility and therefore reduces equilibrium efforts (shifting from the green dashed utility levels to the red dotted utilities). Second, we can reach the maximum effort equilibrium (on the abscissa) in two ways, either through $\frac{r}{b} = \frac{n}{n-1}$ under the winner-take-all prize structure $P^1 = P$ (the solid blue utility level), or through $\frac{n}{n-1} < \frac{r}{b} < \frac{n-1}{\sum_{k=2}^{n} 1/k}$ under an optimal multi-prize structure, i.e. by moving from the dashed green utility level (below the abscissa) up to the red dotted utility level. If $\frac{r}{b} < \frac{n}{n-1}$, the designer should strive to reduce the contestant's equilibrium utility through a more precise ranking, i.e. by increasing r. If $\frac{r}{b} > \frac{n}{n-1}$, the agent's utility in the symmetric equilibrium candidate for a single prize is below her zero effort utility. The designer can then increase the agent's equilibrium utility by either introducing more noise into the success function (reducing r) or by dampening incentives through offering more than one prize. This multiplicity of prizes is dictated, however, not by optimality as such but solely by existence. Finally, the optimal prize structure (4), which enables the designer to collect maximal efforts, consists of no more than three distinct prizes. The highest and lowest prizes, however, may optimally be awarded to multiple players.

Appendix: Proofs

Proof of proposition 1. Denote the probability of player i winning the l^{th} prize among s agents by $f_i^l(e_1, e_2, \ldots, e_s)$. Denote also by $(\hat{e}/\{j_1, j_2, \ldots, j_{l-1}\})$ a vector of efforts of all players other than $\{j_1, j_2, \ldots, j_{l-1}\}$. Then the probability of player i winning prize $l \geq 2$ is given by

$$\begin{aligned} f_i^l(\cdot) &= \sum_{\{j_1,j_2,\ldots,j_{l-1}\}\subseteq\mathcal{N}\backslash\{i\}} \Pr\begin{pmatrix} i \text{ wins } P^l| \\ j_{l-1} \text{ wins } P^{l-1}, \\ \vdots \\ j_1 \text{ wins } P^1 \end{pmatrix} \Pr\begin{pmatrix} j_{l-1} \text{ wins } P^{l-1}| \\ \vdots \\ j_1 \text{ wins } P^1 \end{pmatrix} \cdots \Pr\left(j_1 \text{ wins } P^1\right) \\ &= \sum_{\{j_1,j_2,\ldots,j_{l-1}\}\subseteq\mathcal{N}\backslash\{i\}} f_i^1\left(\hat{e}/\{j_1, j_2, \ldots, j_{l-1}\}\right) f_{j_{l-1}}^1\left(\hat{e}/\{j_1, j_2, \ldots, j_{l-2}\}\right) \cdots f_{j_1}^1\left(\hat{e}\right) \end{aligned}$$

where the sums are taken over all ordered sets of $l-1$ players different from i. Notice that $f_i^l(\cdot)$ only involves simple Tullock winning probabilities $f_j^1(\cdot)$. Since Player i maximizes (1), symmetric equilibrium efforts $\hat{e} = (e, e, \ldots, e)$ satisfy the f.o.c.

$$c'(e) = \sum_{l=1}^{n} \left(\alpha_l(\hat{e}) P^l\right), \ \alpha_l(\hat{e}) = \frac{\partial}{\partial e} f^l(\hat{e}) \tag{5}$$

where we define $\alpha_l(0) = 0$. If the sequence $\alpha_1, \alpha_2, \ldots$ is decreasing in all efforts for a given prize structure, then the symmetric players' utility will be maximized by $P^1 = P$. Coefficients α_1, α_2, and the general α_l are calculated w.l.o.g. for player 1. The α_l^* are the symmetric equilibrium versions.

P^1: For the first prize $\frac{\partial}{\partial e_1}\frac{e_1^r}{\sum e^r}P^1 = \underbrace{\frac{e^r r\,(n-1)\,e_1^{r-1}}{\left((n-1)\,e^r+e_1^r\right)^2}}_{=\alpha_1} P^1$ and for $e_1 = e,\ \alpha_1^* = \frac{1}{e}\frac{(n-1)\,r}{n^2}$.

P^2: $\frac{\partial}{\partial e_1}\left(\frac{e_1^r}{\left((n-2)\,e^r+e_1^r\right)}\frac{(n-1)e^r}{\left((n-1)\,e^r+e_1^r\right)}\right)P^2 = \underbrace{\frac{r\,(n-1)\,e^r e_1^{r-1}\left(e^{2r}\,(n-2)\,(n-1)-e_1^{2r}\right)}{\left((n-2)\,e^r+e_1^r\right)^2\left((n-1)\,e^r+e_1^r\right)^2}}_{=\alpha_2} P^2,$

$$\underset{e_1=e}{\Leftrightarrow}\ \alpha_2^* = \frac{1}{e}\frac{\left(n^2-3n+1\right)r}{(n-1)\,n^2}. \tag{6}$$

P^l: More generally, for prize $l \le n$, we get

$$\frac{\partial}{\partial e_1}\left(\frac{e_1^r}{\left((n-l)\,e^r+e_1^r\right)}\frac{(n-l+1)\,e^r}{\left((n-l+1)\,e^r+e_1^r\right)}\cdots\frac{(n-1)\,e^r}{\left((n-1)\,e^r+e_1^r\right)}P^l\right) =$$
$$\frac{\partial}{\partial e_1}\left(\frac{e_1^r}{\left((n-l)\,e^r+e_1^r\right)}\prod_{x=1}^{l-1}\frac{(n-x)\,e^r}{\left((n-x)\,e^r+e_1^r\right)}P^l\right) =$$
$$\left(\underbrace{\begin{array}{c}\dfrac{r\,(n-l)\,e^r e_1^{r-1}}{\left((n-l)\,e^r+e_1^r\right)^2}\displaystyle\prod_{x=1}^{l-1}\frac{(n-x)\,e^r}{\left((n-x)\,e^r+e_1^r\right)} \\ -\displaystyle\sum_{x=1}^{l-1}\frac{e_1^r}{\left((n-l)\,e^r+e_1^r\right)}\frac{r\,(n-x)\,e^r e_1^{r-1}}{\left((n-x)\,e^r+e_1^r\right)^2}\prod_{y=1,y\neq x}^{l-1}\frac{(n-y)\,e^r}{\left((n-y)\,e^r+e_1^r\right)}\end{array}}_{=\alpha_l}\right)P^l. \tag{7}$$

Using the identity $\prod_{x=1}^{l-1}\frac{(n-x)}{(n-x+1)} = \frac{(n-l+1)}{n}$, we find the symmetric equilibrium expression as

$$\alpha_l^* = \frac{1}{e}\frac{r}{n}\left(\frac{n-l}{n-l+1} - \sum_{k=n-l+2}^{n}\frac{1}{k}\right) \tag{8}$$

which is decreasing in l because the derivative of the first term in parenthesis is negative and, for constant n and r, the sum is increasing for $l > 2$. Notice, moreover, that the last coefficient α_n must be negative since, for $l = n$, the above equals $-\frac{1}{e}\frac{r}{n}\sum_{k=2}^{n}\frac{1}{k}$.[6]

Hence the derivatives of the prize coefficients α are decreasing and $P^1 = P$ induces the contestant's highest equilibrium utilities. Since equilibrium efforts depend on the prize structure, however, it may be the case that equilibrium efforts are higher for some other prize structure (not maximizing the players' utilities). In order to show that this is not the case, we define effort independent coefficients $\beta_l = e\alpha_l$

[6] Thus assigning a positive prize to the contest loser reduces efforts. The same extends to all cases where (8) is negative. Approximately, for large n, $\alpha_l < 0$ when $ln(n) - ln(n-l+2) > \frac{n-l}{n-l+1}$ or roughly $l > \frac{2\mathrm{e}-n+\mathrm{e}n}{\mathrm{e}} \approx 0.63n$ (e is the exponential constant in this footnote).

for $l = 1, \ldots, n$ and obtain the f.o.c.

$$c'(e)e = \sum_{l=1}^{n} \beta_l P^l. \tag{9}$$

Taking equilibrium existence as given, we assume that the s.o.c. holds locally at e^*, i.e.

$$-\frac{1}{e^2}\sum_{l=1}^{n} \beta_l P^l - c''(e) < 0. \tag{10}$$

The solution e^* is then defined implicitly by $\frac{1}{e}\sum_{l=1}^{n} \beta_l P^l = c'(e)$ or $\frac{1}{e}k - c'(e) = 0$, where k is a constant parameter. Therefore

$$\frac{de^*}{dk} = -\frac{\frac{\partial\left(\frac{1}{e}k - c'(e)\right)}{\partial k}}{\frac{\partial\left(\frac{1}{e}k - c'(e)\right)}{\partial e}} = -\frac{\frac{1}{e}}{-\frac{1}{e^2}k - c''(e)} = \frac{e}{k + e^2 c''(e)} \tag{11}$$

and since we assume that the s.o.c. holds at e^*, we know that $k + e^2c''(e) > 0$ and $\frac{e}{k+e^2c''(e)} > 0$. Thus we conclude that if e^* is a solution to player's i maximization problem (i.e. both f.o.c. and s.o.c. hold locally at e^*) then it is also true that e^* is monotonically increasing in k and an increase in $\sum \beta_l P^l$ increases e^*. □

Proof of proposition 2. We show that if all players other than player $i \in \mathcal{N}$ exert effort e^*, then player i's best response is $e = e^*$. Player i's utility is then

$$U(e; e^*) = \frac{e^r}{e^r + (n-1)(e^*)^r}P - ae^b \text{ where } e^* = \left(\frac{1}{ab}\frac{(n-1)r}{n^2}P\right)^{\frac{1}{b}} \tag{12}$$

while, by playing e^*, the player gets $U(e^*; e^*) = \frac{1}{n}P - a\left(e^*\right)^b = \frac{1}{n}\left(1 - \frac{1}{b}\frac{(n-1)r}{n}\right)P$. It is easy to show that both the first and second order conditions hold at e^* when (2) holds.[7] Assume that a critical point x exists where $\frac{d}{de}U(e; e^*)\Big|_{e=x} = 0$.

[7] Deriving $U(e; e^*)$ with respect to e gives $\frac{d}{de}U(e; e^*)$ as

$$\frac{re^{r-1}\left(e^r + (n-1)\left(e^*\right)^r\right) - re^{2r-1}}{\left(e^r + (n-1)\left(e^*\right)^r\right)^2}P - abe^{b-1} = \frac{(n-1)\left(e^*\right)^r re^{r-1}}{\left(e^r + (n-1)\left(e^*\right)^r\right)^2}P - abe^{b-1}.$$

The second derivative gives $\frac{d^2}{de^2}U(e; e^*)|_{e=e^*}$ as

$$\frac{r\left(e^*\right)^{3r-2}(n-1)\left((r-1)(n-1) - (1+r)\right)}{n^3\left(e^*\right)^{3r}}P - ab(b-1)\left(e^*\right)^{b-2} = \frac{r(n-1)}{n^3\left(e^*\right)^2}P\left((n-2)r - nb\right)$$

and $(n-2)r - nb < 0 \Leftrightarrow \frac{r}{b} < \frac{n}{n-2}$ which holds because we require $\frac{r}{b} \leq \frac{n}{n-1} < \frac{n}{n-2}$.

a) We first show that if $0 \leq x < e^*$, then $U(x; e^*) < U(e^*; e^*)$. Derive

$$\frac{d}{de}U(e; e^*) = \frac{(n-1)(e^*)^r r e^{r-1}}{(e^r + (n-1)(e^*)^r)^2} P - abe^{b-1} \tag{13}$$

which equals at a critical point x

$$abx^{b-1} = \frac{(n-1)\,(e^*)^r r x^{r-1}}{(x^r + (n-1)\,(e^*)^r)^2} P. \tag{14}$$

Plugging the critical x^b from (14) into the player's objective (12), we obtain

$$U(x; e^*) = \frac{x^r}{x^r + (n-1)\,(e^*)^r} P - ax^b = \frac{x^r\,(bx^r + (n-1)\,(e^*)^r\,(b-r))}{b\,(x^r + (n-1)\,(e^*)^r)^2} P. \tag{15}$$

Now, $U(x; e^*) < U(e^*; e^*)$ implies that

$$\frac{x^r\,(bx^r + (n-1)(e^*)^r (b-r))}{b\,(x^r + (n-1)(e^*)^r)^2} P < \frac{1}{n}\left(1 - \frac{1}{b}\frac{(n-1)\,r}{n}\right) P \tag{16}$$

which rearranges into

$$((e^*)^r - x^r)\,(n-1)\,(x^r\,(r + bn) + (e^*)^r\,(n-1)\,(bn - (n-1)\,r)) > 0 \tag{17}$$

which is true for $0 \leq x < e^*$ precisely if (2) holds.

b) For $e > e^*$, we proceed to show that $U(e; e^*) - U(e^*; e^*) < 0$ or

$$\frac{e^r}{e^r + (n-1)\,(e^*)^r} P - ae^b - \frac{1}{n}\left(1 - \frac{1}{b}\frac{(n-1)\,r}{n}\right) P < 0. \tag{18}$$

Taking derivatives of $U(e; e^*) - U(e^*; e^*)$ w.r.t. e gives

$$\frac{e^r (e^*)^r (n-1) Pr - abe^b\,(e^r + (e^*)^r (n-1))^2}{e\,(e^r + (e^*)^r (n-1))^2} \tag{19}$$

which is negative—and hence there is no further critical point for $e > e^*$—if

$$\frac{r(n-1)}{abn^2} P < \frac{e^b (e^r + (e^*)^r (n-1))^2}{n^2 e^r (e^*)^r}. \tag{20}$$

Since the l.h.s. equals $(e^*)^b$ this can be rearranged to $n\sqrt{(e^*)^{r+b} e^{r-b}} < e^r + (n-1)(e^*)^r$. Define $h(e) = n\sqrt{(e^*)^{r+b} e^{r-b}}$ and $g(e) = e^r + (n-1)(e^*)^r$—both strictly increasing functions in e. Notice that h and g intersect at $e = e^*$. Moreover, $\frac{d}{de}(g(e) - h(e))$ equals

$$re^{r-1} - \frac{1}{2}(r-b)n(e^*)^{\frac{1}{2}(r+b)}e^{\frac{1}{2}(r-b)-1} > 0 \Leftrightarrow \frac{1}{2r}(r-b)n < \left(\frac{e}{e^*}\right)^{\frac{1}{2}(b+r)}. \tag{21}$$

Since for $r \leq \frac{n}{n-1}b$ the l.h.s is smaller than 1, this is true for all $e > e^*$, thus $g(e) - h(e) > 0$ and (18) holds for all $e > e^*$. As, given any prize structure, the symmetric equilibrium effort is unique, we also have the "only if" part. More precisely, if a symmetric equilibrium exists then $U(e^*; e^*) \geq U(0; e^*)$ only if $\frac{1}{n}\left(1 - \frac{1}{b}\frac{(n-1)\,r}{n}\right)P \geq 0 \;\Leftrightarrow\; r \leq \frac{n}{n-1}b.$ □

Proof of proposition 3. Assume a monotonic prize structure $P^1 \geq P^2 \geq \ldots \geq P^n \geq 0$, $\sum_l P^l = P$, for which a symmetric pure strategy equilibrium inducing positive efforts exists. We claim that if we change prizes to $\left(\frac{1}{n-1}(P - P^n), \ldots, \frac{1}{n-1}(P - P^n), P^n\right)$, then equilibrium efforts decrease, i.e.

$$\frac{1}{n-1}(P - P^n)\sum_{l=1}^{n-1}\beta_l + \beta_n P^n \leq \sum_{l=1}^{n}\beta_l P^l. \tag{22}$$

This is true as we 'shift effort' from the first few prizes—with high weights β—to lower prizes. Formally, there exists an index s, $1 \leq s < n-1$, such that $P^l \geq \frac{1}{n-1}(P - P^n)$ for any $l = 1, \ldots, s$ and $P^l < \frac{1}{n-1}(P - P^n)$ for $l = s+1, \ldots, n-1$.[8] Now,

$$\frac{1}{n-1}(P - P^n)\sum_{l=1}^{n-1}\beta_l + \beta_n P^n \leq \sum_{l=1}^{n}\beta_l P^l \Leftrightarrow \sum_{l=1}^{n-1}\beta_l\left(P^l - \frac{1}{n-1}(P - P^n)\right) \geq 0. \tag{23}$$

Since $\sum_{l=1}^{n-1}\beta_l\left(P^l - \frac{1}{n-1}(P - P^n)\right)$ equals

$$\sum_{l=1}^{s}\beta_l\left(P^l - \frac{1}{n-1}(P - P^n)\right) - \sum_{l=s+1}^{n-1}\beta_l\left(\frac{1}{n-1}(P - P^n) - P^l\right) \tag{24}$$

and

$$\begin{aligned} &\sum_{l=1}^{s}\beta_l\left(P^l - \frac{1}{n-1}(P - P^n)\right) \geq \beta_s\sum_{l=1}^{s}\left(P^l - \frac{1}{n-1}(P - P^n)\right) \text{ while} \\ &\sum_{l=s+1}^{n-1}\beta_l\left(\frac{1}{n-1}(P - P^n) - P^l\right) \leq \beta_s\sum_{l=s+1}^{n-1}\left(\frac{1}{n-1}(P - P^n) - P^l\right), \end{aligned} \tag{25}$$

[8] If $s = n-1$, then this was already the original prize structure and we are done.

we know that

$$\sum_{l=1}^{s}\left(P^l-\frac{1}{n-1}\left(P-P^n\right)\right)-\sum_{l=s+1}^{n-1}\left(\frac{1}{n-1}\left(P-P^n\right)-P^l\right)=0 \tag{26}$$

and thus finally obtain that $\sum_{l=1}^{n-1}\beta_l\left(P^l-\frac{1}{n-1}\left(P-P^n\right)\right)\geq 0$. Since for the original prize structure $U(e^*;e^*)=\frac{P}{n}-c(e^*)\geq P^n=U(0;e^*)$, we have the same inequality for the new prize structure (recall that $c(e)$ is monotonically increasing). For this new prize structure—using the facts that $\sum\beta_l=0$[9] and $P^n\leq\frac{1}{n}P$—we have

$$\begin{aligned}
\frac{P}{n}-c(e^*)\geq P^n \quad &\Rightarrow \quad \frac{P}{n}-\frac{1}{b}\left(\frac{(P-P^n)}{n-1}\sum_{l=1}^{n-1}\beta_l+\beta_nP^n\right)\geq P^n\\
&\Rightarrow \quad \frac{1}{n}\left(1-\frac{1}{b}\frac{n}{n-1}\left(-\beta_n\right)\right)P-\frac{1}{b}\frac{n}{n-1}\beta_nP^n\geq P^n\\
&\Rightarrow \quad \frac{1}{n}\left(1+\frac{1}{b}\frac{n}{n-1}\beta_n\right)P\geq P^n\left(1+\frac{1}{b}\frac{n}{n-1}\beta_n\right)\Rightarrow\left(1+\frac{1}{b}\frac{n}{n-1}\beta_n\right)\geq 0\\
&\Rightarrow \quad \frac{r}{b}\leq\frac{n-1}{\sum_{k=2}^{n}\frac{1}{k}}.
\end{aligned}$$

Assume now that (3) holds. We show that we can find a prize structure 'close' to $(\frac{1}{n}P,\ldots,\frac{1}{n}P)$ for which a symmetric pure strategy equilibrium inducing positive efforts exists. Recall that a given prize structure induces a unique symmetric equilibrium effort that solves (9), i.e.

$$c'(e)e=\sum_{l=1}^{n}\beta_lP^l\Leftrightarrow e^*=\left(\frac{1}{ab}\sum_{l=1}^{n}\beta_lP^l\right)^{\frac{1}{b}}\Leftrightarrow c\left(e^*\right)=\frac{1}{b}\sum_{l=1}^{n}\beta_lP^l \tag{27}$$

where the coefficients β_l are functions of n and r (independent of e and P). Choose a small positive $\varepsilon\leq\frac{1}{n}P$ and consider the prize structure $(\frac{1}{n}P+\frac{1}{n-1}\varepsilon,\ldots,\frac{1}{n}P+\frac{1}{n-1}\varepsilon,\frac{1}{n}P-\varepsilon)$. If a symmetric pure strategy equilibrium exists under this prize structure, then it induces a positive effort of

$$e^*=\left(\frac{1}{ab}\left(\sum_{l=1}^{n-1}\beta_l\left(\frac{1}{n}P+\frac{1}{n-1}\varepsilon\right)+\beta_n\left(\frac{1}{n}P-\varepsilon\right)\right)\right)^{\frac{1}{b}}. \tag{28}$$

Since (3) holds, we indeed get that by exerting an effort of e^* the player achieves a higher utility than what she can achieve by exerting zero effort (while all other players exert $e^*>0$), i.e.

$$U(e^*;e^*)=\frac{P}{n}-c(e^*)\geq\frac{1}{n}P-\varepsilon=U(0;e^*). \tag{29}$$

[9] Since $\sum_{l=1}^{n}\beta_l=\frac{(n-1)r}{n^2}+\sum_{l=2}^{n}\frac{r}{n}\left(\frac{n-l}{n-l+1}-\sum_{k=n-l+2}^{n}\frac{1}{k}\right)=\frac{r}{n}\left(\frac{n-1}{n}+\sum_{k=1}^{n-1}\frac{k-1}{k}-\sum_{k=2}^{n}\frac{k-1}{k}\right)=0.$

This can be shown by expressing $c(e^*) = \frac{1}{b}\left(\sum_{l=1}^{n-1}\beta_l\left(\frac{1}{n}P + \frac{1}{n-1}\varepsilon\right) + \beta_n\left(\frac{1}{n}P - \varepsilon\right)\right)$ and again employing that $\sum \beta_l = 0$. Then (29) is equivalent to $\frac{1}{b}\left(\frac{1}{n-1}(-\beta_n) - \beta_n\right) \le 1$ which, by substituting $\beta_n = -\frac{r}{n}\sum_{k=2}^{n}\frac{1}{k}$ gives $\frac{r}{b}\frac{1}{n-1}\sum_{k=2}^{n}\frac{1}{k} \le 1$ which is true since (3) holds.
To ensure a global maximum we need to show that for every $e \notin \{0, e^*\}$, $U(e;e^*) < U(e^*;e^*)$ where

$$U(e^*;e^*) = \frac{P}{n} - c(e^*) = \frac{P}{n} - \frac{1}{b}\left(\sum_{l=1}^{n-1}\beta_l\left(\frac{1}{n}P + \frac{1}{n-1}\varepsilon\right) + \beta_n\left(\frac{1}{n}P - \varepsilon\right)\right) = \frac{P}{n} - \frac{r}{b}\frac{1}{n-1}\sum_{k=2}^{n}\frac{1}{k}\varepsilon.$$

a) We wish to show that for $e > e^*$, $\frac{d}{de}U(e;e^*) < 0$ implies

$$\frac{dU(e;e^*)}{de} = \left(\frac{n}{n-1}\varepsilon\right)\frac{(n-1)!re^{r-1}\left(e^*\right)^{r(n-1)}}{\prod_{j=1}^{n-1}\left(e^r + (n-j)\left(e^*\right)^r\right)}\left(\sum_{l=1}^{n-1}\frac{1}{\left(e^r + (n-l)\left(e^*\right)^r\right)}\right) - abe^{b-1} < 0. \quad (30)$$

By rearranging and multiplying by $\sum_{k=2}^{n}\frac{1}{k}$ we get

$$\frac{r\varepsilon}{ab(n-1)}\sum_{k=2}^{n}\frac{1}{k} < \frac{e^{b-r}\prod_{j=1}^{n-1}\left(e^r + (n-j)\left(e^*\right)^r\right)\sum_{k=2}^{n}\frac{1}{k}}{n!\left(e^*\right)^{r(n-1)}\left(\sum_{l=1}^{n-1}\frac{1}{\left(e^r+(n-l)\left(e^*\right)^r\right)}\right)} \quad (31)$$

and using the fact that $(e^*)^b = \dfrac{r\varepsilon}{ab(n-1)}\sum_{k=2}^{n}\dfrac{1}{k}$ we obtain

$$\left(e^*\right)^{b+r(n-1)} < \frac{e^{b-r}\prod_{j=1}^{n-1}\left(e^r + (n-j)\left(e^*\right)^r\right)\sum_{k=2}^{n}\frac{1}{k}}{n!\left(\sum_{l=1}^{n-1}\frac{1}{\left(e^r+(n-l)\left(e^*\right)^r\right)}\right)}. \quad (32)$$

Now, for $e > e^*$

$$\frac{e^{b-r}\prod_{j=1}^{n-1}\left(e^r + (n-j)\left(e^*\right)^r\right)\sum_{k=2}^{n}\frac{1}{k}}{n!\left(\sum_{l=1}^{n-1}\frac{1}{\left(e^r+(n-l)\left(e^*\right)^r\right)}\right)} > e^{b-r}\left(e^*\right)^{rn} \quad (33)$$

and

$$\left(e^*\right)^{b+r(n-1)} < e^{b-r}(e^*)^{rn} \Leftrightarrow \left(\frac{e^*}{e}\right)^{b-r} < 1 \quad (34)$$

which is indeed true for $e > e^*$. **b**) For $e < e^*$, showing that $\dfrac{d}{de}U(e;e^*) > 0$ involves exactly the same steps (30)–(34) as under a) for the reversed inequality. □

Proof of proposition 4. We define k as the smallest integer such that the prize structure (4) induces a symmetric equilibrium effort (27) which is smaller or equal to the optimal effort $c^{-1}\left(\frac{P}{n}\right)$. Thus k is the smallest integer such that $\frac{1}{k}\sum_{l=1}^{k}\beta_l \le \frac{b}{n}$. We know that such an integer exists since the l.h.s. is decreasing in k (for $k < n$)

$$\frac{1}{k}\sum_{l=1}^{k}\beta_l \geq \frac{1}{k+1}\sum_{l=1}^{k+1}\beta_l \quad \Leftrightarrow \quad \frac{1}{k}\sum_{l=1}^{k}\beta_l \geq \beta_{k+1} \tag{35}$$

and since β_l is decreasing with l we have $\frac{1}{k}\sum_l \beta_l > \frac{1}{k}\sum_l \beta_k = \beta_k > \beta_{k+1}$ and we establish (35). Moreover, for $k = 1$ we have $\beta_1 = \frac{(n-1)r}{n^2} > \frac{b}{n}$ since (2) is violated, and since (3) holds, we know that, for $k = n-1$, it is true that $\frac{1}{k}\sum_{l=1}^{k}\beta_l \leq \frac{b}{n}$. We thus need to show that given k we can find an $0 \leq \varepsilon < \frac{1}{k}P$ such that the prize structure (4) induces the optimal effort i.e.

$$e^*(k) = \left(\frac{1}{ab}\left(\sum_{l=1}^{k-1}\beta_l\left(\frac{P}{k}+\frac{1}{k-1}\varepsilon\right)+\beta_k\left(\frac{1}{k}P-\varepsilon\right)\right)\right)^{\frac{1}{b}} = \left(\frac{1}{a}\frac{P}{n}\right)^{\frac{1}{b}}. \tag{36}$$

We find ε by ensuring that the utility of the players is minimized and equal to zero, i.e.

$$(k-1)\frac{1}{n}\left(\frac{P}{k}+\frac{1}{k-1}\varepsilon\right)+\frac{1}{n}\left(\frac{P}{k}-\varepsilon\right)-\frac{1}{b}\left(\sum_{l=1}^{k-1}\beta_l\left(\frac{P}{k}+\frac{1}{k-1}\varepsilon\right)+\beta_k\left(\frac{1}{k}P-\varepsilon\right)\right) = 0$$

$$\frac{1}{b}\left(\frac{1}{k-1}\sum_{l=1}^{k-1}\beta_l-\beta_k\right)\varepsilon = \left(\frac{1}{n}-\frac{1}{bk}\sum_{l=1}^{k}\beta_l\right)P.$$

We also know that $\frac{1}{k-1}\sum_{l=1}^{k-1}\beta_l > \frac{b}{n}$ and that $\frac{1}{k}\sum_{l=1}^{k}\beta_l = \frac{1}{k}\sum_{l=1}^{k-1}\beta_l + \frac{1}{k}\beta_k \leq \frac{b}{n}$. Thus

$$\left(\frac{1}{k-1}\sum_{l=1}^{k-1}\beta_l-\beta_k\right) \geq k\left(\frac{1}{k-1}\sum_{l=1}^{k-1}\beta_l-\frac{b}{n}\right) > 0. \tag{37}$$

Therefore we have

$$\varepsilon = \frac{\left(\frac{b}{n}-\frac{1}{k}\sum_{l=1}^{k}\beta_l\right)}{\left(\frac{1}{k-1}\sum_{l=1}^{k-1}\beta_l-\beta_k\right)}P > 0. \tag{38}$$

Finally we need to show that $\varepsilon < \frac{1}{k}P$ implying that

$$\frac{b}{n}-\frac{1}{k}\sum_{l=1}^{k}\beta_l \;<\; \frac{1}{k}\left(\frac{1}{k-1}\sum_{l=1}^{k-1}\beta_l-\beta_k\right) \;\Leftrightarrow\; \frac{b}{n} < \frac{1}{k-1}\sum_{l=1}^{k-1}\beta_l \tag{39}$$

which follows from the definition of k. Using this in (36), we obtain efforts $e^*(k)$ as

$$\left(\frac{1}{ab}\left(\sum_{l=1}^{k-1}\beta_l\left(\frac{1}{k}+\frac{1}{k-1}\frac{\left(\frac{b}{n}-\frac{1}{k}\sum_{l=1}^{k}\beta_l\right)}{\left(\frac{1}{k-1}\sum_{l=1}^{k-1}\beta_l-\beta_k\right)}\right)+\beta_k\left(\frac{1}{k}-\frac{\left(\frac{b}{n}-\frac{1}{k}\sum_{l=1}^{k}\beta_l\right)}{\left(\frac{1}{k-1}\sum_{l=1}^{k-1}\beta_l-\beta_k\right)}\right)\right)P\right)^{\frac{1}{b}}$$
$$=\left(\frac{1}{ab}\frac{b}{n}\left(\frac{\frac{1}{k-1}\sum_{l=1}^{k-1}\beta_l-\beta_k}{\frac{1}{k-1}\sum_{l=1}^{k-1}\beta_l-\beta_k}\right)P\right)^{\frac{1}{b}}=\left(\frac{1}{a}\frac{P}{n}\right)^{\frac{1}{b}}. \qquad \square$$

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