ONLINE APPENDIX

Supplementary Information File associated with "Cycles in Politics: Wavelet Analysis of Political Time-Series"

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Abstract

This is an introduction to the Mathematics of the Continuous Wavelet Transform. We describe some of the most relevant theoretical results, and discuss some of the implementation choices that have to be done in practice. We also briefly describe our Wavelet toolbox, which can be freely downloaded at http://sites.google.com/site/aguiarconraria/joanasoares-wavelets.

The Wavelet

In what follows, $L^2(\mathbb{R})$ denotes the set of square integrable functions, i.e. the set of functions defined on the real line such that $\|x\| := \{\int_{-\infty}^{\infty} |x(t)|^2 dt\}^{1/2} < \infty$, with the usual inner product, $\langle x, y\rangle := \int_{-\infty}^{\infty} x(t)y^*(t)dt$. The asterisk superscript denotes complex conjugation. Given a function $x(t) \in L^2(\mathbb{R})$,

$$X(\xi) := \int_{-\infty}^{\infty} x(t)\, e^{-i2\pi\xi t} dt = \int_{-\infty}^{\infty} x(t)\left[\cos(2\pi\xi t) - i\sin(2\pi\xi t)\right] dt. \tag{A.1}$$

will denote its Fourier transform. We recall the well-known Parseval relation, valid for all $x(t), y(t) \in L^2(\mathbb{R})$, $\langle x(t), y(t)\rangle = \langle X(\xi), Y(\xi)\rangle$, from which the Plancherel identity immediately follows: $\|x(t)\| = \|X(\xi)\|$. The minimum requirements imposed on a function $\psi(t)$ to qualify for being a *mother (admissible or analyzing) wavelet* are that $\psi \in L^2(\mathbb{R})$ and also fulfills a technical condition, usually referred to as the *admissibility condition*, which reads as follows:

$$0 < C_\psi := \int_{-\infty}^{\infty} \frac{|\Psi(\xi)|}{|\xi|} d\xi < \infty, \tag{A.2}$$

where $\Psi(\xi)$ is the Fourier transform of $\psi(t)$,(see Daubechies 1992, 24).

The wavelet ψ is usually normalized to have unit energy: $\|\psi\|^2 = \int_{-\infty}^{\infty} |\psi(t)|^2 dt = 1$. The square integrability of ψ is a very mild decay condition; the wavelets used in practice have much faster decay; typical behavior will be exponential decay or even compact support. For functions with sufficient decay, it turns out that the admissibility condition (A.2) is equivalent to requiring $\Psi(0) = \int_{-\infty}^{\infty} \psi(t)\, dt = 0$. This means that the function ψ has to wiggle up and down the $t-$axis, i.e. it must behave like a wave; this, together with the decaying property, justifies the choice of the term wavelet (originally, in French, *ondelette*) to designate ψ.

The Continuous Wavelet Transform

Starting with a mother wavelet ψ, a family $\psi_{s,\tau}$ of "wavelet daughters"can be obtained by simply scaling ψ by s and translating it by τ

$$\psi_{s,\tau}(t) := \frac{1}{\sqrt{|s|}}\psi\left(\frac{t-\tau}{s}\right), \quad s,\tau \in \mathbb{R}, s \neq 0. \tag{A.3}$$

The parameter s is a scaling or dilation factor that controls the length of the wavelet (the factor $1/\sqrt{|s|}$ being introduced to guarantee preservation of the unit energy, $\left\|\psi_{s,\tau}\right\| = 1$) and τ is a location parameter that indicates where the wavelet is centered. Scaling a wavelet simply means stretching it (if $|s| > 1$), or compressing it (if $|s| < 1$).[1]

Given a function $x(t) \in L^2(\mathbb{R})$ (a time-series), its continuous wavelet transform (CWT) with respect to the wavelet ψ is a function $W_x(s,\tau)$ obtained by projecting $x(t)$, in the L^2 sense, onto the over-complete family $\left\{\psi_{s,\tau}\right\}$:

$$W_x(s,\tau) = \left\langle x, \psi_{s,\tau}\right\rangle = \int_{-\infty}^{\infty} x(t)\frac{1}{\sqrt{|s|}}\psi^*\left(\frac{t-\tau}{s}\right)dt. \tag{A.4}$$

When the wavelet $\psi(t)$ is chosen as a complex-valued function, the wavelet transform $W_x(\tau,s)$ is also complex-valued. In this case, the transform can be separated into its real part, $\mathcal{R}(W_x)$, and imaginary part, $\mathcal{I}(W_x)$, or in its amplitude, $|W_x(\tau,s)|$, and phase, $\phi_x(\tau,s) : W_x(\tau,s) = |W_x(\tau,s)|\, e^{i\phi_x(\tau,s)}$. The phase-angle $\phi_x(\tau,s)$ of the complex number $W_x(\tau,s)$ can be obtained from the formula:

$$\phi_x(\tau,s) = \tan^{-1}\left(\frac{\Im\left(W_x(s,\tau)\right)}{\Re\left(W_x(s,\tau)\right)}\right), \tag{A.5}$$

using the information on the signs of $\Re(W_x)$ and $\Im(W_x)$ to determine to which quadrant the angle belongs to.

For real-valued wavelet functions, the imaginary part is constantly zero and the phase is, therefore, undefined. Hence, in order to separate the phase and amplitude information of a time-series, it is important to make use of complex wavelets. As Lilly and Olhede (2009) explain,

[1]Note that for negative s, the function is also reflected.

analytic wavelets, i.e. wavelets $\psi(t)$ satisfying $\Psi(\xi) = 0$, for $\xi < 0$, are ideal for the analysis of oscillatory signals, since the continuous analytic wavelet transform provides an estimate of the instantaneous amplitude and instantaneous phase of the signal in the vicinity of each time/scale location (τ, s).[2] The importance of the admissibility condition (A.2) comes from the fact that it guarantees that it is possible to recover $x(t)$ from its wavelet transform. When ψ is analytic and $x(t)$ is real, a reconstruction formula is given by

$$x(t) = \frac{2}{C_\psi} \int_0^\infty \left[\int_{-\infty}^\infty \mathcal{R}\left(W_x(s, \tau)\, \psi_{s,\tau}(t) \right) d\tau \right] \frac{ds}{s^2}. \tag{A.6}$$

Therefore, we can easily go from $x(t)$ to its wavelet transform, and from the wavelet transform back to $x(t)$. Note that one can limit the integration over a range of scales, performing a band-pass filtering of the original series. See Daubechies (1992, 27-28) or Kaiser (1994, 70-73) for more details about analytic wavelets.

Localization Properties

Let the wavelet ψ be normalized so that $\|\psi\| = 1$. Also, assume that ψ and its Fourier transform Ψ have sufficient decay to guarantee that the quantities defined below are all finite. We define the center μ_t of ψ by

$$\mu_t = \int_{-\infty}^\infty t \left| \psi(t) \right|^2 dt. \tag{A.7}$$

In other words, the center of the wavelet is simply the mean of the probability distribution obtained from $|\psi(t)|^2$. As a measure of concentration of ψ around its center one usually takes the standard deviation σ_t:

$$\sigma_t = \left\{ \int_{-\infty}^\infty (t - \mu_t)^2 \left| \psi(t) \right|^2 dt \right\}^{\frac{1}{2}}. \tag{A.8}$$

In a total similar manner, one can also define the center μ_ξ and standard deviation σ_ξ of the Fourier transform $\Psi(\xi)$ of ψ.

The interval $[\mu_t - \sigma_t, \mu_t + \sigma_t]$ is the set where ψ attains its "most significant"values, whilst

[2] Note that an analytic function is necessarily complex.

$[\mu_\xi-\sigma_\xi, \mu_\xi+\sigma_\xi]$ plays the same role for $\Psi(\xi)$. The rectangle $[\mu_t - \sigma_t, \mu_t + \sigma_t] \times [\mu_\xi - \sigma_\xi, \mu_\xi + \sigma_\xi]$ in the (t, ξ) $-$plane is called the Heisenberg box or window in the time-frequency plane. We then say that ψ is localized around the point (μ_t, μ_ξ) of the time-frequency plane with uncertainty given by $\sigma_t \sigma_\xi$.

The uncertainty principle, first established by Heisenberg in the context of quantum mechanics, gives a lower bound on the product of the standard deviations of position and momentum for a system, implying that it is impossible to have a particle that has an arbitrarily well-defined position and momentum simultaneously. In our context, the Heisenberg uncertainty principle tells us that there is always a trade-off between localization in time and localization in frequency; in particular, we cannot ask for a function to be, simultaneously, band and time limited. To be more precise, the Heisenberg uncertainty principle establishes that the uncertainty is bounded from below by the quantity $1/4\pi$:

$$\sigma_t \sigma_\xi \geq \frac{1}{4\pi}. \tag{A.9}$$

If the mother wavelet ψ is centered at μ_t, has standard deviation σ_t and its wavelet transform $\Psi(\xi)$ is centered at μ_ξ with a standard deviation σ_ξ, then one can easily show that the daughter wavelet $\psi_{\tau,s}$ will be centered at $\tau + s\mu_t$ with standard deviation $s\sigma_t$, whilst its Fourier transform $\Psi_{s,\tau}$ will have center $\frac{\mu_\xi}{s}$ and standard deviation $\frac{\sigma_\xi}{s}$.

From the Parseval relation, we know that $W_x(s,\tau) = \langle x(t), \psi_{s,\tau}(t) \rangle = \langle X(\xi), \Psi_{s,\tau}(\xi) \rangle$. Therefore, the continuous wavelet transform $W_x(s,\tau)$ gives us local information within a time-frequency window $[\tau + s\mu_t - s\sigma_t, \tau + s\mu_t + s\sigma_t] \times \left[\frac{\mu_\xi}{s} - \frac{\sigma_\xi}{s}, \frac{\mu_\xi}{s} + \frac{\sigma_\xi}{s}\right]$. In particular, if ψ is chosen so that $\mu_t = 0$ and $\mu_\xi = 1$, then the window associated with $\psi_{\tau,s}$ becomes

$$[\tau - s\sigma_t, \tau + s\sigma_t] \times \left[\frac{1}{s} - \frac{\sigma_\xi}{s}, \frac{1}{s} + \frac{\sigma_\xi}{s}\right] \tag{A.10}$$

In this case, the wavelet transform $W_x(s,\tau)$ will give us information on $x(t)$ for t near the instant $t = \tau$, with precision $s\sigma_t$, and information about $X(\xi)$ for frequency values near the frequency $\xi = \frac{1}{s}$, with precision $\frac{\sigma_\xi}{s}$. Therefore, small/large values of s correspond to information about $x(t)$ in a fine/broad scale and, even with a constant area of the windows, $A = 4\sigma_t\sigma_\xi$, their dimensions change according to the scale; the windows stretch for large values of s (broad scales s – low frequencies $\xi = 1/s$) and compress for small values of s (fine scale s – high frequencies $\xi = 1/s$).

Figure A1 illustrates this major advantage afforded by the wavelet transform, when compared to the Short Time Fourier Transform: its ability to perform natural local analysis of a time-series in the sense that the length of wavelets varies endogenously. It stretches into a long wavelet function to measure the low frequency movements; and it compresses into a short wavelet function to measure the high frequency movements.

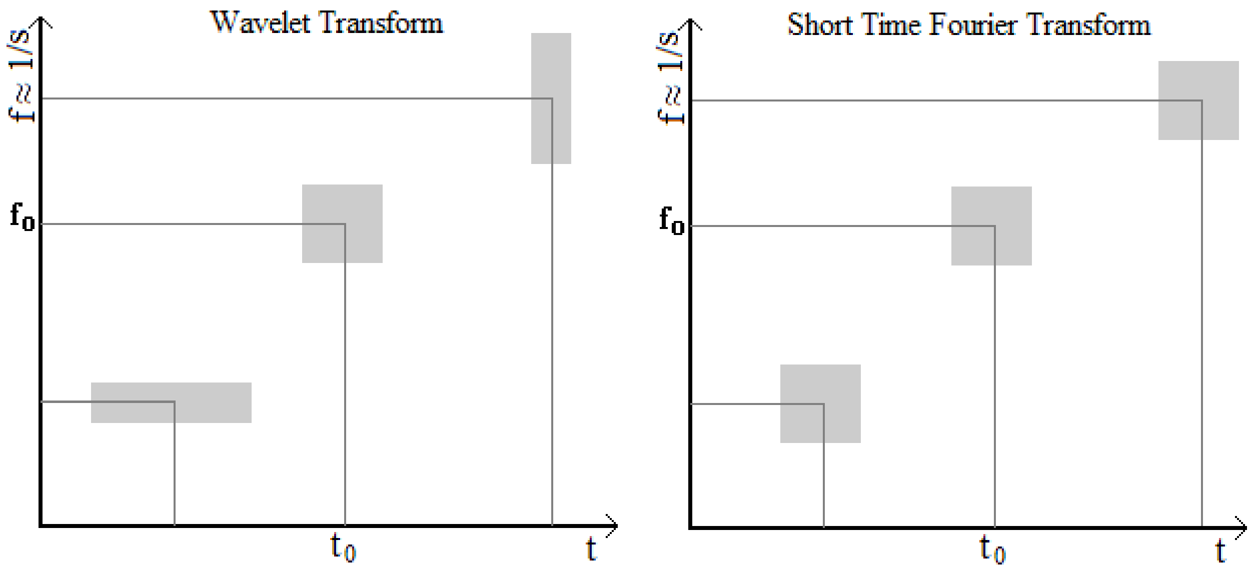

Figure A1: Time-frequency resolution

The Morlet Wavelet: Optimal Joint Time-Frequency Concentration

There are several types of wavelet functions available with different characteristics, such as Morlet, Paull, Cauchy, Mexican hat, Haar, Daubechies, etc. Since the wavelet coefficients $W_x(s,\tau)$ contain combined information on both the function $x(t)$ and the analyzing wavelet $\psi(t)$, the choice of the wavelet is an important aspect to be taken into account, which will depend on the particular application one has in mind. To study cycles, it is important to select a wavelet whose corresponding transform will contain information on both amplitude and phase, and hence, a progressive complex-valued wavelet is a natural choice (the advantage of using a progressive wavelet has already been referred).

We will use the Morlet wavelet, proposed by Goupillaud, Grossman and Morlet (1984):

$$\psi_\eta(t) = \pi^{-\frac{1}{4}} \left(e^{i\eta t} - e^{-\frac{\eta^2}{2}} \right) e^{-\frac{t^2}{2}}. \tag{A.11}$$

The term $e^{-\frac{\eta^2}{2}}$ is introduced to guarantee the fulfillment of the admissibility condition; however, for $\eta \geq 5$ this term becomes negligible. The simplified version

$$\psi_\eta(t) = \pi^{-\frac{1}{4}} e^{i\eta t} e^{-\frac{t^2}{2}} \tag{A.12}$$

of (A.11) is normally used (and still referred to as a Morlet wavelet).

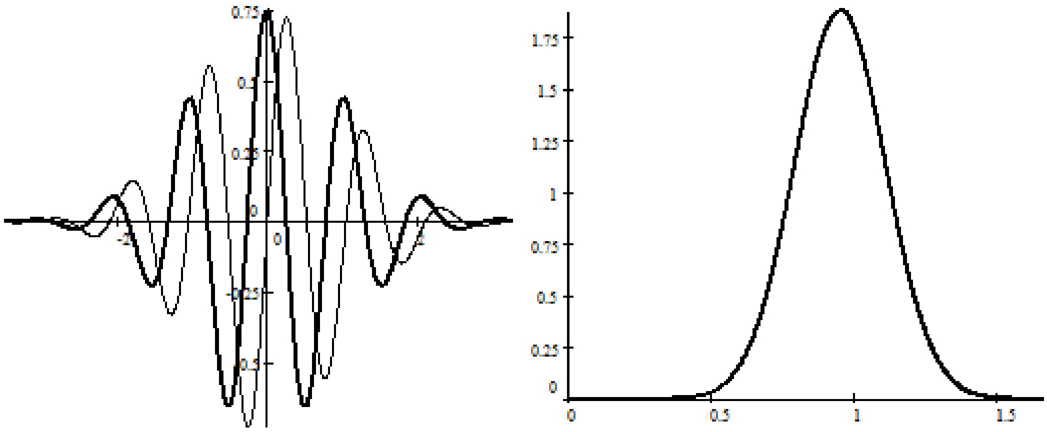

Figure A2: On the left: the Morlet wavelet $\psi_6(t)$ — real part (thick line) and imaginary part (thin line). On the right: its Fourier transform.

This wavelet has interesting characteristics. For $\eta > 5$, for all practical purposes, the wavelet can be considered as analytic; see Foufoula-Georgiou and Kumar (1994).[3] The wavelet (A.12) is centered at the point $\left(0, \frac{\eta}{2\pi}\right)$ of the time-frequency plane; hence, for the particular choice $\eta = 6$, one has that the frequency center is $\mu_\xi = \frac{6}{2\pi}$and the relationship between the scale and

[3] We used $\eta = 6$.in all our computations.

frequency is simply $\xi = \frac{\mu_\xi}{s} \approx \frac{1}{s}$. Thanks to the clear inverse relation between scale and Fourier frequency there is a one-to-one relation between scale and frequency and we will use both terms interchangeably.[4]

It is simple to verify that the time standard deviation is $\sigma_t = 1/\sqrt{2}$ and the frequency standard deviation is $\sigma_\xi = 1/\left(2\pi\sqrt{2}\right)$. Therefore, the uncertainty of the corresponding Heisenberg box attains the minimum possible value $\sigma_t \sigma_\xi = \frac{1}{4\pi}$. In this sense, the Morlet wavelet has optimal joint time-frequency concentration.

Transform of Finite Discrete Data

If one is dealing with a discrete time-series $x = \{x_n,\ n = 0, \ldots, T-1\}$ of T observations with a uniform time step δt, which we can take as the unity ($\delta t = 1$), the integral in (A.4) has to be discretized and is, therefore, replaced by a summation over the T time steps; also, it is convenient, for computational efficiency, to compute the transform for T values of the parameter τ, $\tau = m\delta t;\ m = 0, \ldots, T-1$. In practice, naturally, the wavelet transform is computed only for a selected set of scale values $s \in \{s_k, k = 0, \ldots, F-1\}$ (corresponding to a certain choice of frequencies f_k). Hence, our computed wavelet spectrum of the discrete-time series x will simply be a $F \times T$ matrix W_x whose (k, m) element is given by

$$W_x(k,m) = \frac{1}{\sqrt{s_k}} \sum_{n=0}^{T-1} x_n \psi^* \left((n-m)\frac{1}{s_k} \right) \quad k = 0, \ldots, F-1, \quad m = 0, \ldots, T-1. \tag{A.13}$$

Although it is possible to calculate the wavelet transform using the above formula for each value of k and m, one can also identify the computation for all the values of m simultaneously as a simple convolution of two sequences; in this case, one can follow the standard procedure and calculate this convolution as a simple product in the Fourier domain, using the Fast Fourier Transform algorithm to go forth and back from time to spectral domain; this is the technique

[4] As Meyers, Kelly and O'Brien (1993) say,"for a general wavelet, the relation between scale and the more common Fourier wavelength is not necessarily straightforward; for example, some wavelets are highly irregular without any dominant periodic components. In those cases it is probably a meaningless exercise to find a relation between the two disparate measures of distance. However, in the case of the Morlet wavelet, which is a periodic wavelet enveloped by a Gaussian, it seems more reasonable."

prescribed by Torrence and Compo (1998).

Cone of Influence

As with other types of transforms, the CWT applied to a finite length time-series inevitably suffers from border distortions; this is due to the fact that the values of the transform at the beginning and the end of the time-series are always incorrectly computed, in the sense that they involve missing values of the series which are then artificially prescribed. When using the formula (A.13), a periodization of the data is assumed. However, before implementing formula (A.13), we pad the series with zeros, to avoid wrapping. Because of this zero padding, regions afected by edge effects will under estimate the wavelet power. The region in which the transform suffers from these edge effects is called the cone of influence (COI) and, therefore, its results have to be interpreted carefully.

Wavelet Power Spectrum

In view of the energy preservation formula, and in analogy with the terminology used in the Fourier case, we simply define the (local) wavelet power spectrum as

$$(WPS)_x(s,\tau) = |W_x(s,\tau)|^2, \tag{A.14}$$

which gives us a measure of the local variance.

The seminal paper by Torrence and Compo (1998) is one of the first to give guidance for conducting significance tests for the wavelet power. By using a large number of Monte-Carlo simulations, they derived empirical distributions for the wavelet power corresponding to an AR(0) or a stationary AR(1) process with a certain background Fourier power spectrum (P_ξ), under the null, the corresponding distribution for the local wavelet power spectrum,

$$D\left(\frac{|W_n^x(s)|^2}{\sigma_x^2} < p\right) = \frac{1}{2}P_\xi \chi_v^2,$$

at each time n and scale s. The value of P_ξ is the mean spectrum at the Fourier frequency ξ that corresponds to the wavelet scale s — in our case $s \approx \frac{1}{\xi}$, — and v is equal to 1 or 2, for real or complex wavelets respectively. For more general processes, like an ARMA process, one has to rely on Bootstrap techniques or Monte Carlo Simulations.

Sometimes the wavelet power spectrum is averaged over time for comparison with classical spectral methods. When the average is taken over all times, we obtain the so-called global wavelet power spectrum:

$$GWPS_x(s) = \int |W_x(\tau, s)|^2 \, d\tau. \tag{A.15}$$

Cross-Wavelets

Cross-Wavelet Power and Phase-Difference

The cross-wavelet transform of two time-series, $x(t)$ and $y(t)$, first introduced by Hudgins, Friehe and Mayer (1993), is simply defined as

$$W_{xy}\left(s, \tau\right) = W_x\left(s, \tau\right) W_y^*\left(s, \tau\right), \tag{A.16}$$

where W_x and W_y are the wavelet transforms of x and y, respectively. The cross-wavelet power is given by $|W_{xy}|$. While we can interpret the wavelet power spectrum as depicting the local variance of a time-series, the cross-wavelet power of two time-series depicts the local covariance between these time-series at each scale and frequency. Therefore, cross-wavelet power gives us a quantified indication of the similarity of power between two time-series.

Torrence and Compo (1998) derived the cross-wavelet distribution assuming that the two time-series have Fourier Spectra P_ξ^x and P_ξ^y.Under the null, the cross-wavelet distribution is given by

$$D\left(\frac{\left|W_x W_y^*\right|}{\sigma_x \sigma_y} < p\right) = \frac{Z_v\left(p\right)}{v}\sqrt{P_\xi^x \; P_\xi^y},$$

where $Z_v\left(p\right)$ is the confidence level associated with the probability p for a pdf defined by the square root of the product of two χ^2 distributions. For more general data generating processes one has to rely on Monte Carlo simulations.

The phase difference, $\phi_{x,y}(s,\tau)$, can be computed from the cross-wavelet transform, by using the formula

$$\phi_{x,y}(s,\tau) = \tan^{-1}\left(\frac{\Im\left(W_{xy}(s,\tau)\right)}{\Re\left(W_{xy}(s,\tau)\right)}\right). \tag{A.17}$$

It is possible to show that $\phi_{xy} = \phi_x - \phi_y$,[5] justifying its name. A phase difference of zero indicates that the time series move together at the specified time-frequency; if $\phi_{xy} \in (0, \frac{\pi}{2})$, then the series move in phase, but the time-series x leads y; if $\phi_{xy} \in (-\frac{\pi}{2}, 0)$, then it is y that is leading; a Phase-Difference of π (or $-\pi$) indicates an anti-phase relation; if $\phi_{xy} \in (\frac{\pi}{2}, \pi)$, then y is leading; time-series x is leading if $\phi_{xy} \in (-\pi, -\frac{\pi}{2})$.

With the Phase-Difference, one can also calculate the Instantaneous Time-Lag between the two time-series x and y:

$$(\Delta T)_{xy}(\tau, s) = \frac{\phi_{xy}(\tau, s)}{2\pi\xi(\tau)}, \tag{A.18}$$

where $\xi(\tau)$ is the frequency that corresponds to the scale s.

Wavelet Coherency

As in the Fourier spectral approaches, wavelet coherency can be defined as the ratio of the cross-spectrum to the product of the spectra of both series, and can be thought of as the local correlation, both in time and frequency, between two time-series. The wavelet coherency between two time-series, $x(t)$ and $y(t)$, is defined as follows:

$$R_{xy}\left(s,\tau\right) = \frac{\left|S\left(W_{xy}\left(s,\tau\right)\right)\right|}{\left|S\left(W_{xx}(s,\tau)\right)\right|^{\frac{1}{2}}\left|S\left(W_{yy}(s,\tau)\right)\right|^{\frac{1}{2}}}, \tag{A.19}$$

where S denotes a smoothing operator in both time and scale. Smoothing is necessary. Without that step, coherency is identically one at all scales and times. Smoothing is achieved by a convolution in time and scale. The time convolution is done with a Gaussian and the scale convolution is performed by a rectangular window (see Cazelles et al. 2007 for details). As in the

[5] To be more precise, the above relation holds after we convert $\phi_x - \phi_y$ into an angle in the interval $[-\pi, \pi]$.

case of the traditional (Fourier) coherency, or the (absolute value of the) correlation coefficient, Wavelet Coherency satisfies the inequality $0 \leq R_{xy}(\tau, s) \leq 1$.

Theoretical distributions for wavelet coherency have recently been derived, by Ge (2008), but only when two stationary Gaussian white noises processes are assumed and we use the Morlet wavelet; for more general processes, one again has to rely on Monte Carlo simulation methods.

The Analytic Wavelets Toolbox

Due to its increasing popularity and applicability into a wide range of fields, the amount of wavelet-related software has been growing. Some commercial scientific computing software, such as *Matlab*, now integrate wavelet analysis packages.[6] The reader can find and freely download our toolbox, which runs in MatLab, in http://sites.google.com/site/aguiarconraria/joanasoares-wavelets.

Our toolbox was written with social science applications in mind. To our knowledge, ours is the first toolbox that performs multivariate wavelet analysis, allowing for the possibility of computing multiple and partial wavelet coherencies as well as partial phase-differences. Our toolbox is divided into two folders:

1. **Functions** – containing all the Matlab functions. This has two sub-folders:

    - **Auxiliary** – containing some auxiliary functions to, e.g. generate surrogate series or compute Fourier spectra; it also contains a function to compute measures associated with generalized Morse wavelets.
    - **WaveletTransforms** – containing functions to compute the (analytic) wavelet transform, cross-wavelet transform, wavelet coherency, wavelet phase-difference and time-lag, multiple coherency, partial coherency and partial phase-difference.

[6] E.g. Math Work's *Wavelet Toolbox* for Matlab is one such package. The choice of wavelets is large. The ability to compute the wavelet coherence and cross spectrum was only recently added to the toolbox (Wavelet Toolbox 4.6, released in September 2010). However this toolbox still does not include significance testing, which is a major shortcoming, neither it includes the possibility of performing multivariate wavelet analysis.

2. **Examples** – containing Matlab scripts to generate the pictures associated with each example and application of this paper (and other papers as well). Therefore, it is easy for a research to replicate our results and, then, adjust our codes to his/her own research.

When implementing the transforms, some choices have, naturally, to be made. The most important choice is the wavelet choice. To our knowledge, every economics application of the CWT has made use of the Morlet wavelet. This is our default. Our toolbox also allows for the use of the Generalized Morse Wavelets, which encompass the most popular analytical wavelets (such as the Paul wavelet). Our advice is to use the Morlet. The GMWs can be used for robustness checks. The second most important choice is about significance tests. In our toolbox, the tests of significance are always based on Monte Carlo simulations. The simulations use two different types of methods to construct surrogate series: (1) fitting an ARMA(p, q) model and building new samples by bootstrap or (2) fitting an ARMA(p, q) model and construct new samples by drawing errors from a Gaussian distribution. The 'Econometrics toolbox' is necessary to perform these tests.[7] Our experience tells us that either ARMA(1,0) or ARMA(1,1) fit the data well enough.[8]

The other options are less important. For example, in order to convert frequencies into periods, one has to declare the periodicity of the data. When computing the wavelet coherency, smoothing is necessary, because, otherwise, coherency would be identically one at all scales and times.[9] Smoothing is done by convolution with window functions in time and in frequency. By default, we use the Bartlett window. The other options — Hamming, Hanning, Blackman, etc — require the use of the Signal Processing toolbox. Our experience tells us that the final pictures are quite insensible to this choice.

[7] The user that does not have the Econometrics toolbox can perform significance tests by choosing an ARMA($p, 0$) model with bootstrap. In this case, the model is estimated by OLS and the code is self-contained.

[8] One has to avoid overfitting, otherwise the null may be almost impossible to reject.

[9] The same happens with the Fourier coherency.

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